MQI
09.05.11

Approximation WKB

Séance 11 - ①

1) (a) $-\frac{\hbar^2}{2m}\vec{\nabla}^2\psi(\vec{x}) = (E - V(\vec{x}))\psi(\vec{x})$

$\psi(\vec{x}) = A(\vec{x})\, e^{\frac{i}{\hbar}S(\vec{x})}$

$\nabla\nabla(AB) = 2\nabla A\nabla B + A\nabla^2 B + (\nabla^2 A)B$

$\Leftrightarrow A(\nabla S)^2 - i\hbar A\nabla^2 S - 2i\hbar(\nabla A)(\nabla S) - \hbar^2\nabla A = 2m(E-V)A$

I) Réel: $(\vec{\nabla}S)^2 = 2m(E-V) + \hbar^2(\vec{\nabla}^2 A)/A$

II) Imag: $-\vec{\nabla}^2 S = 2\vec{\nabla}S\cdot\frac{\vec{\nabla}A}{A}$

(b)

$\Leftrightarrow$ II) $-\frac{d^2}{dx^2}S = 2\frac{dS}{dx}\frac{d}{dx}\log A \quad \Big| : \frac{dS}{dx} : 2$

$\Leftrightarrow \frac{1}{2}\frac{1}{\frac{dS}{dx}}\frac{dS^2}{dx^2} + \frac{d}{dx}\log A = 0$

$\Leftrightarrow \frac{1}{2}\frac{d}{dx}\log\frac{dS}{dx} + \frac{d}{dx}\log A = 0$

$\Leftrightarrow \frac{d}{dx}\left(\frac{1}{2}\log\frac{dS}{dx} + \log A\right) = 0$

$\Rightarrow \frac{1}{2}\log\frac{dS}{dx} + \log A = \tilde{C}$

$\Rightarrow \log\left(\sqrt{\frac{dS}{dx}}\cdot A\right) = \tilde{C} \Rightarrow \sqrt{\frac{dS}{dx}}\,A = \underbrace{e^{\tilde{C}}}_{=\tilde{\tilde{C}}} \Rightarrow A = \frac{C}{\sqrt{S'}}$

$$\hbar^2 (d^2A/dx^2)/A \ll (dS/dx)^2$$

(c) ②

$\hbar$ petit (semi-classique), $A(x)$ varie avec $V(x)$, donc lentement

$\Rightarrow$ donc $\frac{d^2A}{dx^2}$ petit

$$\Rightarrow \left(\frac{dS}{dx}\right)^2 = 2m(E - V(x)). \quad \Leftrightarrow \quad \text{Si } \left(\frac{dS}{dx}\right)^2 \gg \hbar \frac{d^2S}{dx^2}$$

$$\Rightarrow \quad \left|\frac{dp}{dx}\right| \ll \frac{1}{\hbar} p^2 = 2\pi \frac{p}{\lambda}$$

(d)

$$\Leftrightarrow \frac{dS}{dx} = \pm\sqrt{2m(E - V(x))}$$

$$\left(\Leftrightarrow dS = \pm dx' \sqrt{2m(E - V(x))}\right)$$

$$\Rightarrow S(x) = \pm \int^x dx' \sqrt{2m(E - V(x))}$$

Avec : $A(x) = \frac{C}{\sqrt{\frac{dS}{dx}}}$, $\frac{dS}{dx} = p(x)$

$$\Rightarrow \psi(x) = \sum_{\pm} \frac{C_\pm}{\sqrt{p(x)}} \exp\left\{\pm i \int dx\, p(x)/\hbar\right\}$$

2) (a)

$$\int dx\, p(x)/\hbar = \int dx \sqrt{2m(-V'x)} = \frac{2}{3} c x^{3/2}, \quad c = (-2mV')^{1/2}/\hbar$$

(b)

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + \psi(x)V(x) = E\psi(x)$$

$$\Leftrightarrow -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = \underbrace{(E - V(x))}_{= V'\cdot x}\psi(x)$$

$$\Leftrightarrow \psi''(x) = \underbrace{-\frac{2m}{\hbar^2}}_{= c^2} V'\cdot x\, \psi(x) \Rightarrow c = \sqrt{\frac{-2mV'}{\hbar^2}}$$

c)

(3) $\psi(x) \propto x^{-1/4} \cos\left(\frac{2c}{3} x^{3/2} - \frac{\pi}{4}\right)$, $x \gg a$

Comme :

1(d) $$\psi(x) = \sum_{\pm} \frac{C_\pm}{\sqrt{p(x)}} e^{\pm \frac{i}{\hbar} \int dx\, p(x)}$$

2(a) $$\int dx\, p(x)/\hbar = \frac{2}{3} c\, x^{3/2}$$

$$\Rightarrow \quad \psi(x) = \frac{C}{\sqrt{p(x)}} \cos\left(\frac{1}{\hbar} \int_a^x dx'\, p(x') - \frac{\pi}{4}\right)$$

$$p(x) = \sqrt{2m(E - Vx)} \sim \sqrt{x}$$

2(d)

④

$$\psi(x) = \frac{C}{\sqrt{p(x)}} \cos\left(\frac{1}{\hbar}\int_a^x dx'\, p(x') - \frac{\pi}{4}\right)$$

$$\psi(x) = \frac{C'}{\sqrt{p(x')}} \cos\left(\frac{1}{\hbar}\int_x^b dx'\, p(x') - \frac{\pi}{4}\right)$$

$\Rightarrow$

$$= \frac{C'}{\sqrt{p(x')}} \cos\left(-\left(\frac{1}{\hbar}\int_x^b dx'\, p(x') - \frac{\pi}{4}\right)\right)$$

5

$$-\int_x^b = \int_b^x = \int_a^x - \int_a^b$$

$$\Rightarrow \quad -\frac{1}{\hbar}\int_x^b dx'\, p(x') + \frac{\pi}{4} = \frac{1}{\hbar}\int_b^x dx'\, p(x') + \frac{\pi}{4}$$

$$= \underbrace{\frac{1}{\hbar}\int_a^x dx'\, p(x') - \left(\frac{1}{\hbar}\int_b^a dx'\, p(x') - \frac{\pi}{4}\right)}$$

$$\stackrel{!}{=} \frac{1}{\hbar}\int_a^x dx'\, p(x') - \frac{\pi}{4} \Rightarrow$$

$$\cos(x) = \cos(x + n\pi)(-1)^n \Rightarrow C' = (-1)^n C$$

$$\Rightarrow \frac{1}{2\pi\hbar}\int_a^b dx\, p(x) = n + \frac{1}{2}$$

$$\cancel{\frac{1}{\hbar}\int_b^a dx\, p(x) - \frac{\pi}{4}}$$

3.)

$$|dp/dx| \ll p^2/\hbar \stackrel{\wedge}{\Rightarrow} \hbar^2 (d^2A/dx^2)/A \ll (dS/dx)^2$$

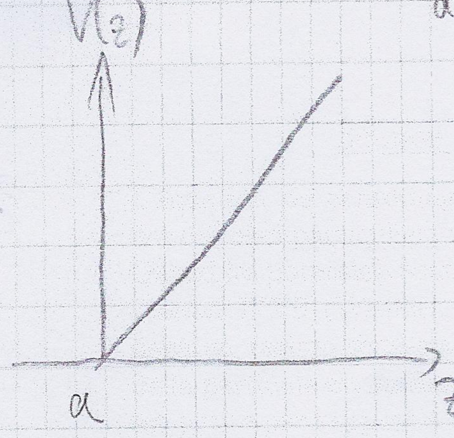
$$A = \frac{C}{\sqrt{\frac{dS}{dx}}} \Rightarrow \frac{dS}{dx} = \frac{C^2}{A^2}$$

$$\Leftrightarrow \quad p = \frac{dS}{dx} = \frac{C^2}{A^2}$$

$$p^2 = \frac{C^4}{A^4} \gg \left|\frac{dp}{dx}\right| = \left|\frac{d^2S}{dx^2}\right|$$

+)

$$\frac{1}{2\pi\hbar}\oint dx\, p(x) = \frac{1}{\pi\hbar}\int_a^b dx\, p(x) = n+\frac{1}{2}$$



$$\Leftrightarrow \frac{1}{2\pi\hbar}\int dz \sqrt{(E_n - V(\;))2m} = n+\frac{1}{2}$$

$$\Leftrightarrow \frac{1}{2\pi\hbar}\sqrt{2m}\int dz\sqrt{E_n - gz} = n+\frac{1}{2}$$

$$\frac{1}{2\pi\hbar}\sqrt{2mg}\int_a^b dx\sqrt{\frac{E_n}{g}-z} = n+\frac{1}{2}$$

$$y = \frac{E_n}{g} - z$$

$$\frac{dy}{dz} = -1 \Rightarrow dy = -dz$$

$$\Leftrightarrow -\frac{\sqrt{2mg}}{2\pi\hbar}\int_{\frac{E_n}{g}-a}^{\frac{E_n}{g}-b} dy\sqrt{y} = -\sqrt{2mg}\left[\frac{2}{3}y^{\frac{3}{2}}\right]_{\frac{E_n}{g}-a}^{\frac{E_n}{g}-b} = n+\frac{1}{2}$$

$$\Rightarrow \; = -\frac{\sqrt{2mg}}{\cancel{2}\pi\hbar}\frac{\cancel{2}}{3}\left[\underbrace{\left(\frac{E_n}{g}-b\right)^{3/2}}_{=0} - \left(\frac{E_n}{g} - \underbrace{a}_{=0}\right)^{3/2}\right] = n+\frac{1}{2}$$

$$E_n = V(b) = g\cdot b$$

$$\Rightarrow \; -\frac{\sqrt{2mg}}{\pi\hbar}\frac{1}{3}\left(\frac{E_n}{g}\right)^{3/2} = n+\frac{1}{2}$$